Maximum hyperchaos in chaotic nonmonotonic neuronal networks

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Hyperchaos in chaotic nonmonotonic neuronal networks is discussed with computer simulations. Maximum chaos with all Lyapunov exponents positive is found not only in the present dissipative model with weak coupling connections between neurons, but also with some strong-coupling connections. Although the model presented is a noninvertible map, the information dimension of simple chaos still yields a good approximation to the Lyapunov dimension. [S1063-651X(97)08107-5]

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I. INTRODUCTION

Neural network models have been extensively investigated with the goal of understanding mechanisms for parallel-distributed information processing [1–4]. Artificial neural networks are composed of simple elements of artificial neurons which aim at modeling biological neurons. In order to study the chaotic characteristics of biological networks, a lot of attention has been focused on chaotic neural networks [5–10]. The characteristic exponent of a chaotic system is the Lyapunov exponent. A high-dimensional chaos with more than one positive Lyapunov exponent is referred to as hyperchaos [11], which implies stretching in two or more directions and therefore leading to more complex dynamical trajectories in phase space. Although a number of chaotic neural networks have been studied, only a few of them have devoted to the discussion of hyperchaos in neural networks. In this paper, hyperchaos in nonmonotonic neuronal networks [9,10] is discussed.

For chaos in dissipative N-dimensional invertible systems, the sums of their Lyapunov exponents are always negative [12,13]. In detail, due to a zero Lyapunov exponent, the number of positive Lyapunov exponents cannot be larger than $N-2$ for diffeomorphisms, while it cannot be larger than $N-1$ for invertible maps. Accordingly, there are some discussions about maximum hyperchaos. Baier and Sahle studied the generalized Rossler equation and pointed out that the maximum hyperchaos for $N=5$ is with three positive Lyapunov exponents, one zero and one negative [14]. Fang found that chaos in the four-dimensional complex-Lorenz-Haken equation can be with three positive Lyapunov exponents [15]. Baier and Klein reported that the maximum hyperchaos in generalized Henon maps possesses $N-1$ positive Lyapunov exponents [16]. One of the potential applications of complex hyperchaos is for cryptographic communication [17].

On the contrary, maximum hyperchaos may result in all Lyapunov exponents being positive for noninvertible dissipative systems. Although we can embed noninvertible equations in a higher-dimensional diffeomorphism or invertible maps [18], it is also important to study them directly. A simple way to construct the maximum hyperchaos map with all Lyapunov exponents positive is to weakly couple one-dimensional chaotic maps [19–21], such as the coupled logistic maps. In this case a chaotic attractor with N positive Lyapunov exponents can be obtained. But if the coupling coefficients in these maps increase a little, one or more Lyapunov exponents will become negative. An interesting question arises and up to now seems not to be discussed: Can strong coupling one-dimensional chaotic maps possess all Lyapunov exponents positive? The main goal of this paper is to give a definite answer to this question. Computer simulations show that nonmonotonic chaotic neuronal models with strong coupling connection between neurons, i.e., with strong couplings, can possess chaos with all Lyapunov exponents positive.

Dimension is one of the basic properties of an attractor. Although the Lyapunov dimension [22] is not a suitable definition for noninvertible maps, simulation results show that for simple chaos in a low-dimensional noninvertible map, the fractal dimension still yields a good approximation to the Lyapunov dimension [23]. In this paper the fractal dimension of hyperchaos is discussed and compared with the Lyapunov dimension.

II. CHAOTIC NEURONAL NETWORK

Now we discuss in detail a chaotic nonmonotonic neuronal network [9] that consists of N analog neurons $\{S_i(t)\}$, $i=1,\ldots,N$, where every neuron S_i is connected to all other neurons S_j by couplings J_{ij}. We use parallel dynamics for the updating of neurons:

$$S_i(t+1)=f(h_i(t+1)), \quad i=1,\ldots,N. \tag{1}$$

Here $h_i(t+1)$ is the weighted input of the ith neuron and is expressed as

$$h_i(t+1)=\sum_{j=1}^{N} J_{ij}S_j(t), \quad i=1,\ldots,N. \tag{2}$$

The activation function is an odd nonmonotonic function

$$f(x)=\tanh(\alpha x)\exp(-\beta x^2). \tag{3}$$

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The input space of the neurons is $(-\infty,\infty)$, while the stable attracting space is within the region $[-1,1]$. The present model becomes the analog Hopfield model if $\beta=0$, and the discrete Hopfield model [1] if $\alpha\rightarrow\infty$ and $\beta=0$. When α and β are large enough, a chaotic attractor can be found in the discrete iterating map (3). Computer simulations show that for a fixed α or β, increasing β or α from zero will force the attractor, changing from fixed points through bifurcation to a periodic attractor and at last to chaos [9].

III. MAXIMUM HYPERCHAOS

Many hyperchaotic systems discussed so far are polynomials in several variables, and in particular, most of them are second-order polynomials [11,14–17,19–21]. In contrast to them, the map studied here is a transcendental function which possesses a stronger nonlinearity than second-order polynomials. Actually, the input $h_i(t+1)$ to the ith neuron is the linear sum of neuron states $S_i(t)$ with linear coupling coefficients J_{ij}. But no matter what values the synaptic connections J are, the relationship between the output $S_i(t+1)$ and the weighted input $h_i(t+1)$ is determined by the chaotic transcendental map (3). The chaotic transcendental map (3) plays a more important role than the linear coupling connection (2). Our computer simulations show that it is easy to find hyperchaos in the model with arbitrary coupling connection matrix J.

As mentioned above, a simple way to construct hyperchaos with all Lyapunov exponents positive is to couple N chaotic neurons and let the couplings between neurons small when compared with their self-feedbacks, i.e., $J_{ii}\gg J_{ij}$ $(i\neq j)$. The coupling coefficients between neurons approaching zero means that each neuron can be regarded as an independent chaotic neuron most of time. Because the probability that the neuronal states fall into the vicinity of zero, only where the coupling effects between neurons need to be considered approaches zero too. Then a chaotic attractor with N positive Lyapunov exponents can be obtained. In this case all the Lyapunov exponents are almost the same and approach the Lyapunov exponent of the one-dimensional discrete iterating map (3).

As a result of its strong nonlinearity, one can expect that hyperchaos with all Lyapunov exponents positive can also exist in networks with strong-coupling connections between neurons. Our computer simulations show that this is true. Consider the neural model with four neurons and the couple synaptic:

$$J=\begin{pmatrix} 0.01 & 1.50 & 0.03 & 0.01\\ 1.60 & 0.01 & 0.10 & 1.00\\ 0.03 & 0.02 & 0.01 & 1.50\\ 0.00 & 1.00 & 1.70 & 0.01 \end{pmatrix}. \tag{4}$$

Figure 1 gives four Lyapunov exponents versus α from 2.0 to 10.0 with $\beta=1.39$ and input stimulus (1, 1, 1, 1). To find all N Lyapunov exponents of the neural model, a set of N linearly independent unit vectors $u_1(t)\wedge\cdots\wedge u_N(t)$ is evolved in the tangent space of the neural dynamics and repeatedly orthonormalized with the Gram-Schmidt orthonormalization procedure. Then the N Lyapunov exponents can be obtained simultaneously from the orthonormalized coefficients [13]. Here 100 000 iterations are performed after the initial 100 000 iterations have been cut.

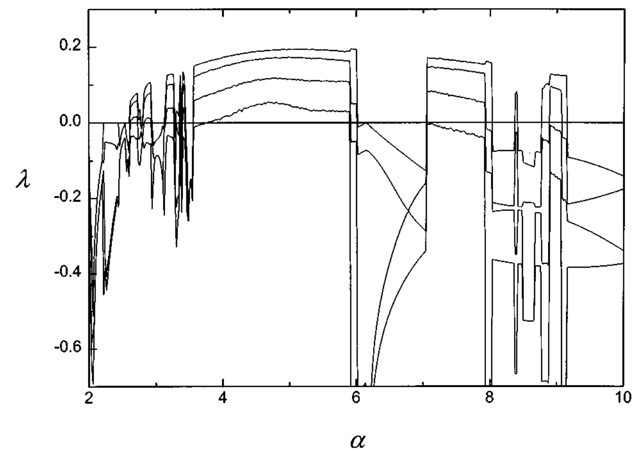

FIG. 1. The four Lyapunov exponents λ versus α from 2.0 to 10.0 with $\beta=1.39$, input (1, 1, 1, 1), and connection (4).

From Fig. 1, we can distinguish four kinds of chaos: first-order chaos with only one positive Lyapunov exponent, second-order chaos with two positive Lyapunov exponents, and similarly for third-order and fourth-order chaos. Fourth-order chaos is actually the maximum hyperchaos in the model. The chaos with all Lyapunov exponents positive means that there is no stable or contracting subspace existing statistically; i.e., all of directions of the attractor are expanding. But this instability does not cause the system trajectories to escape from being confined in a bounded domain. It is the strong nonlinear folding mechanism that keeps chaotic attractor stable. The maximum hyperchaos can be found in the region from 3.78 to 5.96. For example, the four Lyapunov exponents are 0.191, 0.167, 0.109, and 0.033 (±0.002) for fourth-order chaos with $\alpha=5.60$; 0.193, 0.172, 0.115, and 0.045 with $\alpha=5.00$; and 0.173, 0.148, 0.087, and 0.018 with $\alpha=4.00$. The third-order chaos can be found in [3.56, 3.76] or [7.06, 7.92], and the four Lyapunov exponents are 0.156, 0.133, 0.056, and -0.034 with $\alpha=7.80$. The second-order chaos can be found in [5.92, 6.00] or [8.90, 8.06], and the first-order chaos in [8.78, 8.88] or [9.08, 9.14].

On the $\alpha,-\beta$ plane with $1.0\leqslant\alpha\leqslant8.0$ and $1.0\leqslant\beta\leqslant6.0$, the regions with different kinds of attractors are shown in Fig. 2. In the figure, the number n represents an nth-order

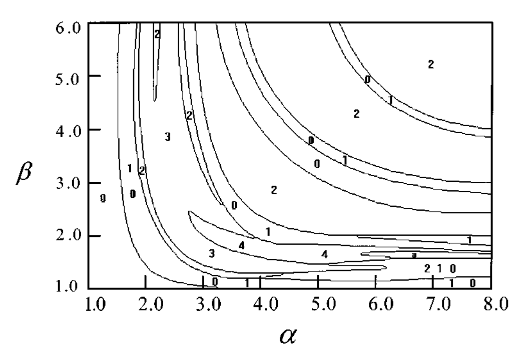

FIG. 2. The regions with different kinds of attractors of the α-β plane with $1.0\leqslant\alpha\leqslant8.0$ and $1.0\leqslant\beta\leqslant6.0$. Here 0 represents periodic attractor; n (>0) represents nth-order chaos.

attractor. Here 10 000 iterations are performed after the initial 10 000 iterations omitted. Figure 2 shows that most chaos are second-order hyperchaos. The maximum chaos mainly concentrates in the range of $2.8 \leqslant \alpha \leqslant 6.5$ and $1.3 \leqslant \beta \leqslant 2.3$.

From Figs. 1 and 2, one can see that in a very narrow range of bifurcation parameters, fast transitions from periodic attractor, first-order or second-order chaos to fourth-order chaos, or from periodic attractor or first-order chaos to third-order chaos often occur. However, for a high-dimensional chaotic system, the distribution of Lyapunov exponents is often considered as a smooth function of bifurcation parameters [21]. There should be a natural transition route from first-order chaos via second-order hyperchaos to third-order hyperchaos and at last to fourth-order hyperchaos. Actually, our simulations show that when the computational precision is high enough, the natural transition route appears gradually. For example, from Fig. 1 one can see that there is a fast transition from periodic attractor to third-order chaos when α is changed from 3.54 to 3.56. But if the calculating step is down to 10^{-12}, a number of transitions between the nth-order attractor and $(n+1)$th-order attractor can be found.

Moreover, computer simulations show that it is easy to find hyperchaos in models with arbitrary coupling connection matrix J, but not so easy to find a strong-coupling connection J for models possess chaos with all Lyapunov exponents positive.

IV. FRACTAL DIMENSIONS OF HYPERCHAOS

A chaotic attractor often possesses a fractal dimension. For the high-dimensional invertible maps, we can determine its fractal dimension along the lines of the box-counting algorithm: Let a fiducial volume with side length ϵ evolve with time. Each axis of the fiducial volume will be scaled by a factor proportional to that direction's Lyapunov number, i.e., $\exp(\lambda_i)$. The direction with positive Lyapunov exponent will be expanded, thereby contributing an integer for fractal dimension.

By this means the Lyapunov dimension [22] is obtained and normally approaches to capacity dimension. A precise formulation is that the Lyapunov dimension is equal to the information dimension if the probability measures are smooth along unstable directions [24,25].

However, in this paper, the Lyapunov dimension is not true in general as the map is noninvertible. Therefore we calculate the capacity dimension and the information dimension of the chaos. To achieve this, the phase space is divided into N-dimensional boxes of side r. An initial point is chosen, and the map is iterated a sufficient number of times that the subsequently generated points can be considered to be on the attractor. A list is made of those boxes containing how many points on the attractor. From the number $n(r)$ of the boxes that contain at least one point and from the entropy $H(r)$, the capacity and information dimensions can be fitted out respectively [26]. One example is given in Fig. 3 for fourth-order hyperchaos of the neural model with $\alpha=3.0$, $\beta=2.0$, and connections in Eq. (4). Here the capacity dimension $D_c=3.27(\pm 0.05)$ and information dimension $D_I=3.26$ are obtained. The data was achieved from 200 000 points after the initial transient of 10 000 points have been cut with input stimulus (1, 1, 1, 1).

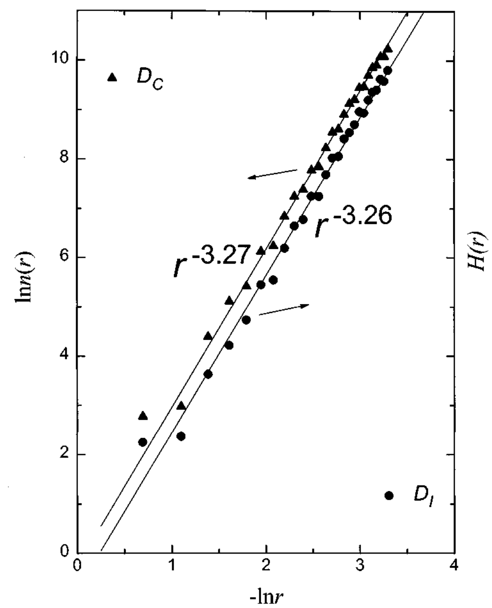

FIG. 3. The logarithm of box-counting numbers ln$n(r)$ and the entropy $H(r)$ via logarithmic scale ln$r$ for the fourth-order hyperchaos in neural model with $\alpha=3.0$, $\beta=2.0$, and connection (4).

In Table I, the capacity dimension D_c and information dimension D_I are calculated for different kinds of chaotic attractors of the neural model with synaptic connection (4). For a comparison, the Lyapunov dimensions D_L are also given, although it is not an appropriate concept. Apart from the computational error, one can see from the table that the capacity dimension is equal to the information dimension. And one can also reach the following conclusion [23]: Even

TABLE I. Comparison of capacity and information dimensions with Lyapunov dimension of chaos.

α	β	Sign of Lyapunov exponents	D_L (± 0.005)	D_c (± 0.05)	D_I (± 0.05)
1.8	3.5	0 − − −	1.000	0.99	0.99
11.0	12.0	+ − − −	2.184	2.17	2.17
5.0	5.0	+ + − −	3.026	2.87	2.85
5.0	9.0	+ + − −	3.134	3.13	3.12
5.0	1.2	+ + + −	4.0	2.97	2.97
2.5	3.0	+ + + −	4.0	3.53	3.53
4.0	1.5	+ + + +	4.0	3.05	3.04
3.0	2.0	+ + + +	4.0	3.27	3.26

for simple chaos of a noninvertible high-dimensional map, such as the first-order chaos, the information dimension still yields a surprisingly good approximation to the Lyapunov dimension. However, for high-order chaos, our computer simulation results show that there does not exist any simple relationship between the Lyapunov exponents and the information dimension. A set of more positive Lyapunov exponents does not imply a larger fractal dimension for complex hyperchaos in the noninvertible map. For example, the fractal dimension of the third-order hyperchaos with $\alpha=2.5$ and $\beta=3.0$ is larger than that of fourth-order one with $\alpha=3.0$ and $\beta=2.0$ or with $\alpha=4.0$ and $\beta=1.5$; the fractal dimension of second-order hyperchaos with $\alpha=5.0$ and $\beta=9.0$ is larger than that of the fourth-order one with $\alpha=4.0$ and $\beta=1.5$.

V. CONCLUSION

In this paper we show that hyperchaos can easily be found in the neural network model with nonmonotonic activation function (3). The hyperchaos with all Lyapunov exponents positive is found not only in our dissipative model with a weak-coupling connection between neurons, but also with some strong-coupling connections. There is a natural transition route from first-order chaos via second-order chaos to third-order chaos and then to fourth-order chaos. The maximum hyperchaos, i.e., fourth-order chaos, can be applied to cryptographic communication [17]. The capacity dimension of hyperchaos equals to its information dimension. Even for simple chaos of a noninvertible map, the information dimension still yields a surprisingly good approximation to the Lyapunov dimension. But a set of more positive Lyapunov exponents does not imply a larger fractal dimension for complex hyperchaos in a noninvertible map.

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